B.Math.(Hons.) IIIrd year
Second semestral exam 2013
Algebraic Number Theory
Instructor — B.Sury
ANSWER ANY FIVE — Max. Marks 50

Q 1.(10 marks)
Prove that the class group of $(\sqrt{-10})$ has order 2.

OR

Let K be a number field of degree 3 over $\mathbf{Q}$ such that $-49 \leq disc(K) < 0$. Show that O_K is a PID.

Q 2.(12 marks)
Consider a cyclotomic field $\mathbf{Q}(\zeta)$ where ζ is a primitive $(2^n + 1)$-th root of unity. If $\Phi(X)$ denotes the minimal polynomial of ζ over $\mathbf{Q}$, consider the reduction $\overline{\Phi}(X)$ of $\Phi(X)$ mod 2 in $(\mathbf{Z}/2\mathbf{Z})[X]$. Determine the number of irreducible factors of the polynomial $\overline{\Phi}(X)$ and their degrees.

OR

Show that the discriminant of the polynomial $X^n + aX + b$ is
$(-1)^{n(n-1)/2}(n^n b^{n-1} + (1-n)^{n-1}a^n)$.
Use this to deduce that the ring of integers in $\mathbf{Q}(\alpha)$ is $\mathbf{Z}[\alpha]$, where α satisfies $X^3 + X + 1$.

Q 3.(10 marks)
Let A be a Dedekind domain. If A is a UFD, then prove that A must be a PID.

OR

Let A be a Dedekind domain. If A contains only finitely many prime ideals, prove that A is a PID.

Q 4.(11 marks)
Let d be a square-free positive integer congruent to 1 mod 4. Let b be the smallest positive integer such that $db^2 \pm 4$ is a perfect square, say a^2, with $a > 0$. Prove that the fundamental unit of $\mathbf{Q}(\sqrt{d})$ is $a + b\sqrt{d}$.

OR

Let K be a number field. Prove that an element of O_K is a unit if and only if its norm over $\mathbf{Q}$ is ± 1. Further, show that there is no unit of norm -1 in $\mathbf{Q}(\sqrt{79})$.

Q 5.(10 marks)
Let K be a number field. For a prime number p, write $pO_K = P_1^{e_1} \cdots P_g^{e_g}$ where P_i are prime ideals. Prove that O_K/P_i has cardinality p^{f_i} for some positive integers $f_1, \cdots, f_g$ and that

$$e_1 f_1 + \cdots + e_g f_g = [K : \mathbf{Q}]$$

OR

If $p \equiv 1 \bmod 4$ is a prime number, determine the splitting of $2O_K$ where $K = \mathbf{Q}(\sqrt{p})$.

Q 6.(11 marks)
Let L/K be a Galois extension of number fields. Let Q be a non-zero prime ideal of O_L and $P = Q \cap O_K$. Define the decomposition group of Q and prove that it maps onto the Galois group of the residue field O_L/Q over O_K/P.

OR

Consider $L := \mathbf{Q}(\zeta_{31})$, where ζ_{31} is a primitive 31-st root of unity. Let K be a subfield of L which has degree 3 over $\mathbf{Q}$. Prove that $O_K \neq \mathbf{Z}[\alpha]$ for any α.

Q 7.(10 marks)
Define the discriminant of a number field K. Determine with proof the sign of disc K. Further, describe the embedding of O_K as a lattice in $\mathbf{R}^n$ where $[K : \mathbf{Q}] = n$ and determine its volume.

OR

Let K be a number field. Prove that there exists a $\mathbf{Q}$-basis $\{\alpha_1, \cdots, \alpha_n\}$ of K which is contained in O_K.
Further, show that each element of O_K is expressible as $\frac{1}{d}\sum_{i=1}^{n} m_i\alpha_i$ where $d = disc(\alpha_1, \cdots, \alpha_n)$, m_i are integers such that $d|m_i^2$ for each $i \leq n$.

Q 8.(10 marks)
Prove that $(1-\zeta_p)^{p-2}$ is the different ideal of $\mathbf{Q}(\zeta_p)$ where p is a prime, and ζ_p is a primitive p-th root of unity.

OR

Prove that the class number of any number field is finite. State the results you use.